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W-T identities and a candidate "droplet" Lagrangean for the Ising spin glass

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Abstract. In search for a microscopic description of "droplet-like" properties for the Ising spin glass (single component order parameter, zero modes i.e. correlation functions vanishing at infinity) we reconsider the two-packet model of Bray and Moore, which is effectively Replica-Symmetric and enjoys zero modes but only up to one-loop. We show how an appropriate change in the limits of the basic parameters of the model (packet size and replica number) allows for a derivation of Ward–Takahashi (WT) identities, thus ensuring the existence of zero modes to all orders and opening the way for a Lagrangean formulation of a "droplet-like" field theory for the Ising spin glass.

PACS. 64.70.Pf Glass transitions – 64.60.Cn Order-disorder transformations; statistical mechanics of model systems

Spin glass theory has presented so far the schizophrenic aspect of two conflicting approaches difficult to reconcile. The so-called mean-field like approach developed around Parisi [1–3] solution of the Sherrington-Kirkpatrick [4] mean-field model uses standard field theory (mean-field, loops, renormalisation group, WT identities, ...) on a Lagrangean built with replicated fields. It is a "microscopic" theory for the spin glass, in the same sense as the ϕ^4 theory for the ferromagnet. The alternative droplet-like theory of Fisher and Huse [5] and Bray and Moore [6], despite abundant results, does not appeal to such a Lagrangean field theory starting point and is far removed from "microscopic" description. So there is a strong motivation to search better into the replicated ϕ^3 Lagrangean and examine whether the characteristic features for a "droplet" like theory could fit in.

If one were to put up one, it would have the features of a theory with an effective Replica-Symmetric order parameter (a ferromagnetic in disguise). It would also possess zero modes to allow for an algebraic decrease at large distances of its correlations. But such features were indeed present, years before the birth of droplet theory, in an ansatz proposed by Bray and Moore [7] (BM), were replica symmetry was broken into "two-packets", and restored at the very end. From their results, calculated at one-loop, it could be checked that both features mentioned above (no RSB, zero modes) were present. However there was no guarantee that the zero modes would survive beyond one loop (indeed they would not and it was suggested that a generalization to multi-packet could help zero modes to remain massless).

In this note we wish to take a new look at the "two-packets" RSB, as BM applied it in [7] to a spin glass Lagrangean with cubic coupling. We wish first to understand why this RSB did not give rise to WT identities that would have protected the zero modes to all orders. Indeed such identities have been derived [8] in the framework of a Lagrangean field theory for systems with R steps of RSB (with in the end the Parisi limit $R \to \infty$). One essential ingredient in the proof is the selection of "infinitesimal permutations", in fact infinitesimal like $1/R$. In the two-packet theory, one packet has m replicas ($a\ b\ c$ ...) and the other $n-m$ ($\alpha\ \beta\ \gamma\ \ldots$). In the end n is set to zero (as it should for replicas) and m sent to infinity. Infinitesimal permutations are then easily selected (they are associated with transverse generators) and they go to zero with $1/m$. So following the same steps as in [8], we identify below where the model fails to yield WT identities. Bringing the appropriate change allows then WT identities to be established, thereby giving life to a "droplet" Lagrangean field theory for the spin glass.

1 Framework for WT identities

We start with a permutation invariant free-energy functional

$$F\{Q_{A,B}\} = F\{Q_{PA,PB}\} \tag{1}$$

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A is a replica index that could have been denoted (i, a) $i = 1$ or 2, but is more economically replaced by a or α, the roman-greek notation of BM. The order parameter Q_{AB} is given by the stationarity condition on equation (1) and as in [7], at mean-field level one has

$$Q_{ab} = Q_1 = Q\frac{m-n}{m-n/2} \tag{2a}$$

$$Q_{\alpha\beta} = Q_2 = Q\frac{m}{m-n/2} \tag{2b}$$

$$Q_{a\alpha} = Q_0 = Q\frac{m}{m-n/2}\left[1-\frac{n}{m}+\frac{n}{m^2}\right]^{1/2}. \tag{2c}$$

On the other hand, from the definition of Legendre transform one has[1]

$$W\{H_{AB}\} + F\{Q_{AB}\} = \sum_{(AB)} H_{AB}Q_{AB} \tag{3}$$

where H_{AB} is an external (unphysical) conjugate source, and hence

$$\frac{\partial F\{Q_{CD}\}}{\partial Q_{AB}} = H_{AB}, \tag{4}$$

yielding stationarity when the source is set to zero.

The invariance under permutation then writes

$$\frac{\partial F}{\partial Q_{AB}}\{Q_{CD} + \delta Q_{CD}\} = H_{AB} + \delta H_{AB} \tag{5}$$

where

$$Q_{PA;PB} = Q_{AB} + \delta Q_{AB}. \tag{6}$$

Now we have to make a choice for P (a choice amounting to use transverse generators). Just like in [8] we can divide packet one into m/p equal bunches of p roman replicas and packet two into $(n-m)/p$ equal bunches of greek replicas. The permutation chosen is for example, exchanging the first bunch of p roman replicas $(a_1\ b_1\ c_1\ ...)$ with the first bunch of p greek replicas $(\alpha_1\ \beta_1\ \gamma_1\ ...)$. In the following we keep $a\ b\ c$ ... or $\alpha\ \beta\ \gamma$ ... for replicas that do no belong to the exchanged first bunches.

Let us now look at the effect of such a permutation P by computing δQ_{CD} (or δH_{AB}). Clearly one has

$$\delta Q_{ab} = \delta Q_{\alpha\beta} = \delta Q_{a\alpha} = \delta Q_{a_1\alpha_1} = 0 \tag{7a}$$

$$\delta Q_{a_1b_1} = -\delta Q_{\alpha_1\beta_1} = Q_2 - Q_1 \equiv \delta Q_0 \tag{7b}$$

$$\delta Q_{a_1\alpha} = -\delta Q_{\alpha_1\alpha} = Q_2 - Q_0 \equiv \delta Q_2 \tag{7c}$$

$$\delta Q_{a\alpha_1} = -\delta Q_{a_1a} = Q_1 - Q_0 \equiv \delta Q_1 \tag{7d}$$

and all these quantities are infinitesimal with $1/m$. So that we are entitled to expand equation (5) in δQ_{CD} and keep only the first term in its Taylor expansion, provided the resulting summation does not ruin the infinitesimality. Thus from equation (5) we obtain, the formal WT identity

$$\sum_{CD}\frac{\partial^2 F}{\partial Q_{AB}\partial Q_{CD}}\delta Q_{CD} = \delta H_{AB}. \tag{8}$$

Note that this relationship has zero on its RHS for (AB) as in equation (7a), or a non-zero RHS in δH for (AB) as in equations (7b, 7c, 7d).

2 Some notations

To write out in detail equation (8) we need to introduce some notation for

$$\frac{\partial^2 F}{\partial Q_{AB}\partial Q_{CD}} \equiv M^{AB;CD}. \tag{9}$$

Noting the *overlaps* $A \cap B$

$$a \cap b = 1 \tag{10a}$$

$$\alpha \cap \beta = 2 \tag{10b}$$

$$a \cap \alpha = 0 \tag{10c}$$

the matrix $M^{AB;CD}$ will be identified by its *overlaps* $A \cap B$ and $C \cap D$ written as upper indices. To have a complete set of matrix elements we need also to specify how many maximal *cross-overlaps* we have: 0 if $AB \neq CD$, 1 if $A = C$, or $B = C$, or $A = D$, or $B = D$, and 2 if $A = C$, $B = D$ or $A = D$, $B = C$. This closeness index we write as a heavy lower index. For example we have

$$M^{ab;ab} = M^{1;1}_{\mathbf{2}} \tag{11a}$$

$$M^{ab;ac} = M^{1;1}_{\mathbf{1}} \tag{11b}$$

$$M^{ab;cd} = M^{1;1}_{\mathbf{0}}. \tag{11c}$$

Alike for the $1 \longleftrightarrow 2$ exchange in the upper indices. Also

$$M^{a\alpha;a\alpha} = M^{0;0}_{\mathbf{2}} \tag{12a}$$

$$M^{a\alpha;b\beta} = M^{0;0}_{\mathbf{0}}. \tag{12b}$$

The only ambiguity left is to distinguish $M^{a\alpha;\alpha b}$ from $M^{\alpha a;a\beta}$ which we write

$$M^{a\alpha;\alpha b} = M^{0;0}_{\mathbf{1}(2)} \tag{13a}$$

$$M^{\alpha a;a\beta} = M^{0;0}_{\mathbf{1}(1)} \tag{13b}$$

exhibiting in parenthesis whether the repeated replica is roman (1) or greek (2).

[1] For reasons of convenience we have changed the notation of BM with their $Q_3 \to Q_1$, $Q_2 \to Q_0$, $Q_1 \to Q_2$.

3 WT identity for AB = $a_1 b$

We carry out explicitly the $\sum_{CD}$ summation. We have

$$\left(M^{a_1b;a_1b}+\sum_c M^{a_1b;a_1c}+\sum_{b_1} M^{a_1b;b_1b}+\sum_{b_1c} M^{a_1b;b_1c}\right)(-\delta Q_1)+\left(\sum_\alpha M^{a_1b;a_1\alpha}+\sum_{\alpha b_1} M^{a_1b;b_1\alpha}\right)(\delta Q_2)+\left(\sum_{\alpha_1} M^{a_1b;\alpha_1b}+\sum_{\alpha_1c} M^{a_1b;\alpha_1c}\right)(\delta Q_1)+\left(\sum_{\alpha_1\beta} M^{a_1b;\alpha_1\beta}\right)(-\delta Q_2)=-\delta H_1. \quad (14)$$

Expliciting the summations and using the above notations (11)–(13) we get

$$\left[M_{\mathbf{2}}^{1;1}+(m-p-1)\,M_{\mathbf{1}}^{1;1}+(p-1)\,M_{\mathbf{1}}^{1;1}+(p-1)\,(m-p-1)\,M_{\mathbf{0}}^{1;1}\right]\delta Q_1-\left[(n-m-p)\,M_{\mathbf{1}(1)}^{1;0}+(p-1)\,(n-m-p)\,M_{\mathbf{0}}^{1;0}\right]\delta Q_2-\left[pM_{\mathbf{1}(1)}^{1;0}+p\,(m-p-1)\,M_{\mathbf{0}}^{1;0}\right]\delta Q_1+\left[p\,(n-m-p)\,M_{\mathbf{0}}^{1;2}\right]=\delta H_1. \quad (15)$$

The first observation is that if we keep the limits $n\to 0$, followed by $m\to\infty$ as taken in [7], the LHS of equation (15) contains factors going to infinity and it is no longer justified to replace equation (5) by the first term in its Taylor expansion equation (8). If we want to get rid of the terms in m in equation (15) we can choose

$$m\equiv n^{1/2}\mu \quad (16)$$

and with $n\to 0$ first, and then $\mu\to\infty$. Note that a choice $m=n^\alpha\mu$, for $\alpha>1/2$ would sent Q_0 to infinity. As for the choice $\alpha<1/2$ it would imply $Q_1=Q_2=Q_0=Q$, leaving no room for the identities looked after. With the special choice $\alpha=1/2$, in the limit $n=0$ we have equation (2) replaced by

$$Q_1=Q_2=Q \quad (17a)$$

$$Q_0=Q+\frac{Q}{2\mu^2} \quad (17b)$$

and equation (7) by

$$\delta Q_0=0 \quad (18a)$$

$$\delta Q_1=\delta Q_2=-\frac{Q}{2\mu^2}\equiv\delta Q. \quad (18b)$$

Thus for μ large we have an infinitesimal transform. (Note that with Eq. (18a) we did not bother to write terms in δQ_0 that occur in Eq. (14)).

With $n=0$ and with the choice of (16) we now get

$$\left[M_{\mathbf{2}}^{1;1}-2M_{\mathbf{1}}^{1;1}+M_{\mathbf{0}}^{1;1}\right]\delta Q+p^2\left[-M_{\mathbf{0}}^{1;1}+\left(2M_{\mathbf{0}}^{1;0}-M_{\mathbf{0}}^{1;2}\right)\right]\delta Q=\delta H. \quad (19)$$

Here p, the number of replicas in the exchanged bunch can be any finite integer, $p>1$. So clearly, if we wish an unambiguous answer, it would be necessary that from other equations the coefficient of p^2 be fixed equal to zero.

4 Related WT identities for AB = $a_1 b_1$ and AB = ab

Both identities have a vanishing RHS, a_1b_1 leads to δH_0 (vanishing as in Eq. (18a)) and ab leads to zero (as in Eq. (7a)). The calculation follows exactly the same line as in the previous section. Spelling out the two equations obtained yields (i) the vanishing of the coefficient of p^2 in equation (19), giving the diagonal component $M_{\mathbf{0}}^{1;1}$ in terms of the off-diagonal components $M_{\mathbf{0}}^{1;0}$, $M_{\mathbf{0}}^{1;2}$; (ii) the corresponding relationship for $M_{\mathbf{1}}^{1;1}$ (see below).

5 WT identities exhibited

With the above we can now express WT identities obtained for $A\cap B=1$:

Replicon for overlap 1:

$$M_{\mathbf{2}}^{1;1}-2M_{\mathbf{1}}^{1;1}+M_{\mathbf{0}}^{1;1}=\frac{\delta H}{\delta Q} \quad (20a)$$

and

$$M_{\mathbf{0}}^{1;1}=2M_{\mathbf{0}}^{1;0}-M_{\mathbf{0}}^{1;2} \quad (20b)$$

$$M_{\mathbf{1}}^{1;1}=M_{\mathbf{1}}^{1;0}+M_{\mathbf{0}}^{1;0}-M_{\mathbf{0}}^{1;2}. \quad (20c)$$

In exactly the same way one gets corresponding equations for $A\cap B=2$

$$M_{\mathbf{2}}^{2;2}-2M_{\mathbf{1}}^{2;2}+M_{\mathbf{0}}^{2;2}=\frac{\delta H}{\delta Q} \quad (21a)$$

$$M_{\mathbf{0}}^{2;2}=2M_{\mathbf{0}}^{2;0}-M_{\mathbf{0}}^{1;2} \quad (21b)$$

$$M_{\mathbf{1}}^{2;2}=M_{\mathbf{1}}^{2;0}+M_{\mathbf{0}}^{2;0}-M_{\mathbf{0}}^{1;2}. \quad (21c)$$

Finally taking $A\cap B=0$, one gets

$$M_{\mathbf{2}}^{0;0}-\left(M_{\mathbf{1}(1)}^{0;0}+M_{\mathbf{1}(2)}^{0;0}\right)+M_{\mathbf{0}}^{0;0}=\frac{\delta H}{\delta Q} \quad (22a)$$

$$M_{\mathbf{0}}^{0;0}=\frac{1}{2}\left(M_{\mathbf{0}}^{1;0}+M_{\mathbf{0}}^{2;0}\right) \quad (22b)$$

$$M_{\mathbf{1}(1)}^{0;0}=M_{\mathbf{1}}^{1;0}+\frac{1}{2}\left(M_0^{2;0}-M_{\mathbf{0}}^{1;0}\right) \quad (22c)$$

$$M_{\mathbf{1}(2)}^{0;0}=M_{\mathbf{1}}^{2;0}-\frac{1}{2}\left(M_{\mathbf{0}}^{2;0}-M_{\mathbf{0}}^{1;0}\right). \quad (22d)$$

6 Effect of a magnetic field

At mean-field level it is easily verified that the equations of motion that yield equation (2) are proportional to the equation giving the lowest eigenvalue (Replicon) of the Hessian. That is, the WT identity for the Replicon zero mode is trivially checked at the mean-field level. In the presence of an external magnetic field (distinct from the unphysical conjugate fields H_{AB}) the equations of motion yielding the order parameters is unchanged but for an extra term H. This H cannot appear in the Hessian, a second derivative matrix, since in the Lagrangean it occurs in the linear term $H \sum_{AB} \phi_{AB}$, with $Q_{AB} = \langle \phi_{AB} \rangle$. Hence the presence of an external magnetic field suppresses the Goldstone modes and hence the transition, just like it occurs in $O(N)$ systems.

7 A return on mean-field

Let us look back at equations (2) which are only valid at the mean-field level. Actually, in our derivation, we only have used a milder form of equations (18). To get the WT identities we only needed

$$\delta Q_0 = 0 \tag{23a}$$

$$\delta Q \sim 1/\mu^2 \tag{23b}$$

for the limits

$$n = 0 \tag{24a}$$

$$\frac{1}{\mu^2} \lll 1. \tag{24b}$$

Is equation (23) valid to all orders beyond mean-field? This is easily checked to all orders in the paramagnetic phase. Going back to equations (3, 4), we have the order parameter Q_{AB} defined by

$$Q_{AB} = \partial W\{H\} / \partial H_{AB} \tag{25}$$

that is by

$$Q_{AB} = \frac{1}{\mathcal{N}} \int \prod_{(CD)} D\phi_{CD}\ \phi_{AB} \tag{1}$$

$$\times \exp\left\{\mathcal{L}\{\phi\} + \sum_{(CD)} H_{CD}\phi_{CD}\right\} \tag{26a}$$

$$\mathcal{N} = \int \prod_{(CD)} D\phi_{CD}\ \exp\left\{\mathcal{L}\{\phi\} + \sum_{(CD)} H_{CD}\phi_{CD}\right\}. \tag{26b}$$

Here $\mathcal{L}\{\phi\}$ is the cubic BM Lagrangean, where the fields $\phi_{AB}(i)$ are coupled via $w/6 \sum_i \sum_{ABC} \phi_{AB}(i)\, \phi_{BC}(i)\, \phi_{CA}(i)$. We have everywhere omitted the space (site) dependence since, in the end, the external source $H_{AB}(i)$ is always taken as H_{AB}, site independent.

Consider then the perturbation expansion of equations (25, 26) giving Q_{AB} as a power series of H_{CD}. If we choose $H_{CD} = H$, we then have $Q_{AB} = Q$ and we write it as

$$Q = f(H)\, H. \tag{27}$$

If we choose now $H_{ab} = H_{\alpha\beta} = H$ and $H_{a\beta} = H_0$, one then has (no H_0 dependence when n and m vanish)

$$Q_{ab} = Q_{\alpha\beta} \equiv Q = f(H)\, H. \tag{28}$$

One also has (because $W(H, H_0)$ can only be even in H_0)

$$Q_{a\beta} \equiv Q_0 = g(H; H_0)\, H_0 \tag{29}$$

and from equation (27) when $H_0 = H$,

$$g(H; H) = f(H). \tag{30}$$

Hence for $H_0 - H \sim 1/\mu^2$, we have

$$Q_0 = \left[f(H) + \frac{\partial g(H;H)}{\partial H_0}(H_0 - H) + \dots\right] H_0 \tag{31}$$

and

$$Q - Q_0 = \left[f(H) + H\frac{\partial g(H;H)}{\partial H_0}\right](H - H_0) + \mathcal{O}(H - H_0)^2. \tag{32}$$

That is, under the limits equation (24), one gets

$$\delta Q \sim \delta H, \tag{33}$$

thus justifying equation (23).

8 Comments and conclusion

We have thus exhibited the Goldstone behavior for the three Replicon modes (20a, 21a, 22a), these modes acquiring a mass proportional to the (unphysical) conjugate field δH, with in the end $\delta H = 0$. A detailed examination of the Hessian matrix components shows [9] besides that both the anomalous and longitudinal modes with zero overlap (as in Eq. (22)) also remain massless.

Altogether we have 10 relationships (for 15 components). Note that with the five off-diagonal components one builds the seven diagonal terms $M_{\mathbf{0}}^{i;i}$, $M_{\mathbf{1}}^{i;i}$ with $i = 1$, 2, 0 as in equations (20bc), (21bc), (22bcd). The last three diagonal terms $M_{\mathbf{2}}^{i;i}$ (the one that contain the kinetic term for non-zero value of the momentum) complete the setup.

To conclude we have given the right to exist to a spin glass theory whose starting point is formally identical to Bray and Moore two-packets theory but with the crucially different limits for the parameters n, m, namely with

$$m \equiv n^{1/2}\mu. \tag{16}$$

With the limit $n = 0$, $\mu \to \infty$ we have then derived ten relationships between the fifteen components of the two-point (one particle irreducible) functions. Relationships between three-point functions could be derived in the same way all these relationships being *non-perturbative*.

This new Lagrangean is a good candidate to describe "droplet" aspects of the Ising spin glass theory. It raises many questions and enjoys the following properties:

(i) as in BM, the order parameter is, in the end, Replica Symmetric, a disguised ferromagnet;
(ii) its associated free-energy is probably worse (lower) than the Parisi free-energy in the vicinity of the upper critical dimension. What would be the effect of dimension (that enters via loops) and would there be a critical dimension below which the "droplet" description would prevail is a crucial question to investigate;
(iii) its correlation functions enjoy several Goldstone modes. These modes become massive in the presence of an external magnetic field. They interact through cubic coupling. Thus their effective coupling cannot vanish in the infra-red like is the case for $O(N)$ systems. It is thus expected that the $1/p^2$ behavior of the Goldstone modes will develop anomalies. Large distance anomal behavior of correlation functions (computed for example in $6-D$ dimension) will then have to be confronted with numerically obtained droplet exponents.

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[2] With equation numbering having been messed up at the editing stage, the reader should rather consult the cond-mat version.